J. Math. Anal. Appl. 365 (2010) 806–808

Contents lists available at ScienceDirect

Journal of Mathematical Analysis and Applications

www.elsevier.com/locate/jmaa

Note

On the regularity of generalized MHD equations

Luo Yuwen

School of Mathematics & Physics, Chongqing University of Technology, Chongqing 400050, PR China

ARTICLE INFO

Article history:
Received 8 July 2009
Available online 24 October 2009
Submitted by J. Guermond

Keywords:
Generalized magneto-hydrodynamics equations
Regularity conditions

ABSTRACT

This paper studies the regularity of generalized magneto-hydrodynamics equation on the condition $0 < \alpha = \beta < 3/2$. It will show if $\nabla u \in L^{p,q}$ on $[0, T)$ with

$$\frac{2\alpha}{p} + \frac{3}{q} \leqslant 2\alpha, \qquad \frac{3}{2\alpha} < q \leqslant +\infty,$$

then the solution can be extended past T.

In this paper, we study the regularity of 3D generalized magneto-hydrodynamics (GMHD) equations

$$\begin{cases} \partial_t u + u \cdot \nabla u = -\nabla p + b \cdot \nabla b - (-\Delta)^\alpha u, \\ \partial_t b + u \cdot \nabla b = b \cdot \nabla u - (-\Delta)^\beta b, \\ \nabla \cdot u = 0, \quad \nabla \cdot b = 0, \\ u(0, x) = u_0(x), \quad b(0, x) = b_0(x), \end{cases} \tag{1}$$

where u is the velocity field, b is the magneto field and p is a scalar pressure, u_0, b_0 are the initial velocity and magneto field respectively, with $\nabla \cdot u_0 = 0$, $\nabla \cdot b_0 = 0$. The fractional power of Laplace operator $(-\Delta)^\alpha$ is defined as

$$\cap(-\Delta)^\alpha f = |\xi|^{2\alpha} \hat{f}(\xi).$$

For notational convenience, we write $\Lambda = (-\Delta)^{1/2}$.

To author's best knowledge, all existing regularity results for (1) are for α or β no less than $3/4$. In [3], J. Wu dealt with the case $\alpha > 1/2 + d/4$ and $\beta > 1/2 + d/4$, where d is the dimension of R^n; Y. Zhou [5] studied the regularity of (1) in the case $1 \leqslant \alpha = \beta \leqslant 3/2$. He also considered the cases $\alpha = \beta \geqslant 5/4$, $\alpha \geqslant 5/2 - \beta$ together with $1 \leqslant \beta \leqslant 5/4$; for the case $3/4 < \alpha = \beta < 5/4$, G. Wu studied it in [2].

We also mention others class of regularity criteria for 3D MHD equations. For regularity in term of pressure, see [6]. In [8], Zhou and Fan established regularity criteria only in terms of the pressure. Zhou and Gala [7] established regularity criteria in the multiplier space.

In this paper, the author will prove the regularity criteria under the condition $0 < \alpha = \beta \leqslant 3/2$. The main result is:

Theorem 1. *Let $0 < \alpha < 3/2$. Assume that the initial velocity and magneto field satisfy $u_0, v_0 \in H^1(R^3)$. If on $(0, T)$, $\nabla u \in L^{p,q}$ with*

$$\frac{2\alpha}{p} + \frac{3}{q} \leqslant 2\alpha, \qquad \frac{3}{2\alpha} < q \leqslant +\infty,$$

then the solution of (1) can be extended past T.

E-mail address: petitevin@gmail.com.

0022-247X/$ – see front matter
doi:10.1016/j.jmaa.2009.10.052

Proof. Multiplying the first equation of (1) by Δu and the second equation by Δb, integrating by parts and adding the results, we have

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\int_{R^3}\left(\|\Lambda u\|_{L^2}^2+\|\Lambda b\|_{L^2}^2\right)dx+\left\|\Lambda^{\alpha+1}u\right\|_{L^2}^2+\left\|\Lambda^{\alpha+1}b\right\|_{L^2}^2&=\int_{R^3}\partial_k u_i\cdot\partial_i u_j\cdot\partial_k u_j\,dx+\int_{R^3}\partial_k u_i\cdot\partial_i b_j\cdot\partial_k b_j\,dx\\&\quad-\int_{R^3}\partial_k b_i\cdot\partial_i b_j\cdot\partial_k u_j\,dx-\int_{R^3}\partial_k b_i\cdot\partial_i u_j\cdot\partial_k b_j\,dx\\&\triangleq I_1+I_2+I_3+I_4.\end{aligned}\tag{2}$$

First, we assume $q<\infty$. By the Hölder inequality, interpolation inequality, Sobolev imbedding theorem and Young's inequality, we get

$$\begin{aligned}|I_1|&=\left|\int_{R^3}\partial_k u_i\cdot\partial_i u_j\cdot\partial_k u_j\,dx\right|\leqslant\|\nabla u\|_{L^q}\|\nabla u\|_{L^\gamma}^2\leqslant C\|\nabla u\|_{L^q}\|\nabla u\|_{L^2}^{2\theta}\|\nabla u\|_{L^r}^{2(1-\theta)}\\&\leqslant C\|\nabla u\|_{L^q}\|\nabla u\|_{L^2}^{2\theta}\left\|\Lambda^{\alpha+1}u\right\|_{L^2}^{2(1-\theta)}\leqslant\left\|\Lambda^{\alpha+1}u\right\|_{L^2}^2+C\|\nabla u\|_{L^q}^{1/\theta}\|\nabla u\|_{L^2}^2,\end{aligned}\tag{3}$$

where q,γ,r,θ satisfy

$$\begin{cases}\dfrac{1}{q}+\dfrac{2}{\gamma}=1,\\[2mm]\dfrac{1}{\gamma}=\dfrac{\theta}{2}+\dfrac{1-\theta}{r},\quad 2<\gamma<r,\\[2mm]r=\dfrac{2n}{n-2\alpha}=\dfrac{6}{3-2\alpha}.\end{cases}$$

From these equations, we have

$$\theta=1-3/(2\alpha)+3/(\gamma\alpha).\tag{4}$$

Similarly, we have

$$|I_2+I_3+I_4|\leqslant\left\|\Lambda^{\alpha+1}b\right\|_{L^2}^2+C\|\nabla u\|_{L^q}^{1/\theta}\|\nabla b\|_{L^2}^2,\tag{5}$$

where θ is the same as (4). So if $2\alpha/p+3/q\leqslant 2\alpha$, then $1/\theta\leqslant p$ and

$$\int_0^T\|\nabla u\|_{L^q}^{1/\theta}\,dt<+\infty.$$

Combining (2), (3), (5), one obtains

$$\frac{1}{2}\frac{d}{dt}\left(\|\nabla u\|_{L^2}^2+\|\nabla b\|_{L^2}^2\right)\leqslant C\left(\|\nabla u\|_{L^q}^{1/\theta}\left(\|\nabla u\|_{L^2}^2+\|\nabla b\|_{L^2}^2\right)\right).\tag{6}$$

By the standard Gronwell inequality, we conclude that the solution can be continued after $t=T$ in the case $p<+\infty$.

When $q=+\infty$, then by Hölder inequality, the first term of (2) in the right is bounded by $\|\nabla u\|_{L^\infty}\|\nabla u\|_{L^2}^2$ and the others are bounded by $\|\nabla u\|_{L^\infty}\|\nabla b\|_{L^2}^2$. The condition $2\alpha/p+3/q\leqslant 2\alpha$ implies that $p\geqslant 1$ when $q=\infty$, then

$$\int_0^t\|\nabla u\|_{L^\infty}\,dt<+\infty$$

holds. By the standard Gronwell inequality, the conclusion of this theorem holds. This completes the proof. □

Remark 1. When $\alpha=1$, the condition of Theorem 1 becomes

$$\frac{2}{p}+\frac{3}{q}\leqslant 2,\qquad \frac{3}{2}\leqslant q\leqslant+\infty.$$

This coincides with the result of Zhou [4], He and Xin [1] for MHD equations.

Remark 2. By the Biot–Savart law and the bounds of the Riesz transforms on L^p spaces $(1 < p < +\infty)$, one obtains

$$\|\nabla f\|_{L^p} \leqslant C\|\operatorname{curl} f\|_{L^p}, \quad \forall 1 < p < \infty. \tag{7}$$

So we can derive the regularity in terms of vorticity from Theorem 1. That is, under the same assumption for α, β and initial value of Theorem 1, if $\omega_0 = \operatorname{curl} u_0 \in L^2(R^3)$, $\omega = \operatorname{curl} u \in L^{p,q}$ with

$$\frac{2\alpha}{p} + \frac{3}{q} \leqslant 2\alpha, \qquad \frac{3}{2\alpha} < q < \infty,$$

then the solution remains smooth on $(0, T]$.

References

[1] C. He, Z. Xin, On the regularity of weak solutions to the magneto-hydrodynamic equations, J. Differential Equations 213 (2005) 235–254.
[2] G. Wu, Regularity criteria for the 3D generalized MHD equations in terms of vorticity, Nonlinear Anal. 71 (2009) 4251–4258.
[3] J. Wu, Generalized MHD equations, J. Differential Equations 195 (2003) 284–312.
[4] Y. Zhou, Remarks on regularities for the 3D MHD equations, Discrete Contin. Dyn. Syst. 12 (2005) 881–886.
[5] Y. Zhou, Regularity criteria for the generalized viscous MHD equations, Ann. Inst. H. Poincaré Anal. Non Linèaire 24 (2007) 491–505.
[6] Y. Zhou, Regularity criteria for the 3D MHD equations in terms of the pressure, Internat. J. Non-Linear Mech. 41 (2006) 1174–1180.
[7] Y. Zhou, S. Gala, Regularity criteria for the solutions to the 3D MHD equations in the multiplier space, Z. Angew. Math. Phys. (2009), doi:10.1007/s00033-009-0023-1, in press.
[8] Y. Zhou, J. Fan, On regularity criteria in terms of pressure for the 3D viscous MHD equations, submitted for publication.